Double Pendulum Power

Method for Extracting Power from a Mechanical Oscillator

-

A Numerical Analysis using the Runge Kutta Method to Solve the Euler Lagrange Equation for a Double Pendulum with Mechanical Load

Anon Ymous, M.Sc. M.E.

anon.ymous.dpp@gmail.com

2013-12-28

Abstract

The power of a double pendulum can be described as the power of the oscillations at the fixed pivot expressed as torque and angular velocity with the following equations:

$$\tau_{PivotTorque} = m_{OscMass} r^2_{RadiusOfOsc} \ddot{\theta}_{AngAcc} \tag{1}$$

$$P_{PivotPower} = m_{OscMass} r^2_{RadiusOfOsc} \ddot{\theta}_{AngAcc} \dot{\theta}_{AngVel} \tag{2}$$

We conclude that this power can be made useful, and has been done so in a number of experiments (for example Milkovic covered in this report). The Runge Kutta numerical method is used to solve the Euler Lagrange equations for evaluation and simulation. Euler Lagrange elegantly describes the force equilibriums and exchange of kinetic energy between the pendulum masses. Mechanical load is then applied to the system as a torque at the fixed pivot.

Without load, this is a pretty straight forward analysis. What we find interesting is the magnitude of the reactive power in relation to the initial kinetic energy of the system. However, when we add load as a counter velocity torque on the fixed pivot we get very interesting results.

$$\tau_{load} = \mu_{FactorOfInertia} m_{OscMass} r_{RadiusOfOsc} \dot{\theta}_{AngVel} \tag{3}$$

When we extract torque as a function of inertia (velocity) it will of course reduce angular acceleration, which in turn will reduce velocity and consequently, kinetic energy. However our simulations show that the system will transfer torque back to the outer pendulum and eventually reach an equilibrium and resonance at any load where no kinetic energy is lost.

From these simulations we conclude that it is possible to extract considerably more energy from a double pendulum system than is used to set the outer pendulum in motion initially. This is due to the fact that rotation of the outer pendulum creates what we might call *artificial gravity*, i.e. a constantly oscillating force acting as a torque on the fixed pivot.

1 The Physics

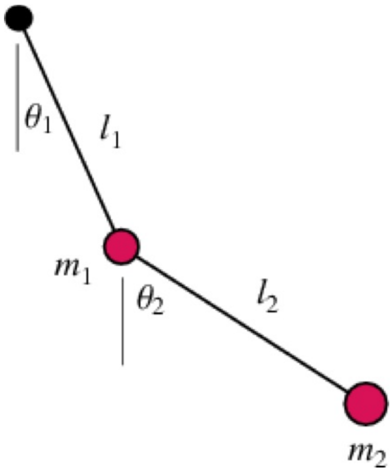

Figure 1: double pendulum

First consider the well known Euler Lagrange equations for the double pendulum.

$$(m_1+m_2)l_1\ddot{\theta}_1+m_2l_2\ddot{\theta}_2cos(\theta_1-\theta_2)+m_2l_2\dot{\theta}_2^2sin(\theta_1-\theta_2)+g(m_1+m_2)sin\theta_1 = 0 \quad (4)$$

$$m_2l_2\ddot{\theta}_2 + m_2l_1\ddot{\theta}_1cos(\theta_1-\theta_2) - m_2l_1\dot{\theta}_1^2sin(\theta_1-\theta_2) + m_2gsin\theta_2 = 0 \quad (5)$$

To be able to solve these equations using the Runge Kutta method we elaborate the equations for the angular accelerations respectively: $\ddot{\theta}_1$ and $\ddot{\theta}_2$.

$$\begin{aligned}\ddot{\theta}_1 =& \frac{-g(2m_1+m_2)sin\theta_1 - m_2gsin(\theta_1-2\theta_2)}{l_1(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))} \\ &- \frac{2sin(\theta_1-\theta_2)m_2(\dot{\theta}_2^2l_2+\dot{\theta}_1^2l_1cos(\theta_1-\theta_2))}{l_1(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))}\end{aligned} \quad (6)$$

$$\ddot{\theta}_2 = \frac{2sin(\theta_1-\theta_2)(\dot{\theta}_1^2l_1(m_1+m_2)+g(m_1+m_2)cos\theta_1+\dot{\theta}_2^2l_2m_2cos(\theta_1-\theta_2))}{l_2(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))} \quad (7)$$

If we have a look at the Pivot Power we see that it is an oscillating power that has a negative sign when velocity and acceleration are not aligned. In plain words it means that we can not extract the Pivot Power directly when it is negative. What we can do however is to extract the kinetic energy of the velocity, which in turn will influence acceleration of both pendulum masses according to Euler Lagrange.

First we examine the system without load. The Runge Kutta method is used to solving the equation numerically. We use the following input data: $m_1 = 1\text{kg}$, $m_2 = 0.1\text{kg}$, $l_1 = 0.2\text{m}$, $l_2 = 0.1\text{m}$, $\dot{\theta}_2 = 314$ rad/s (i.e. the initial speed of rotation of the outer pendulum is 50 Hz which equals an initial kinetic energy of $E_k = 50\text{J}$).

We also assume rotation is in the horizontal level so that $g = 0$ and that there is no friction. As we can se in the graph the outer pendulum is constantly transferring energy to the inner pendulum mass by use of centrifugal force and angular acceleration and vice versa. The result is that the the both the pendulum masses are constantly accelerating/decelerating (oscillating), without any more input of energy.

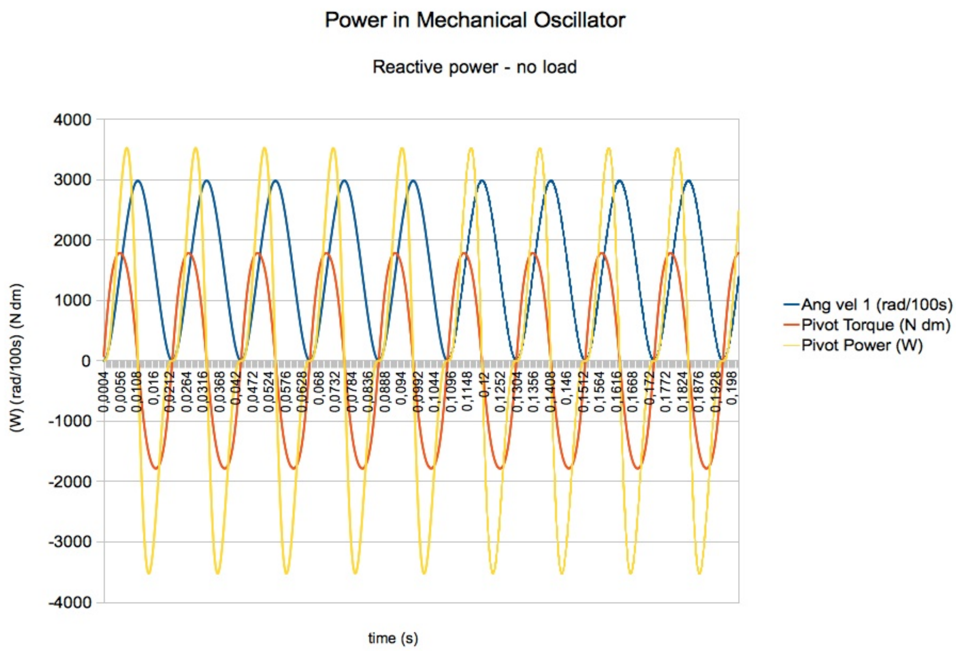

Figure 2: Torque, velocity and reactive power at the fixed pivot

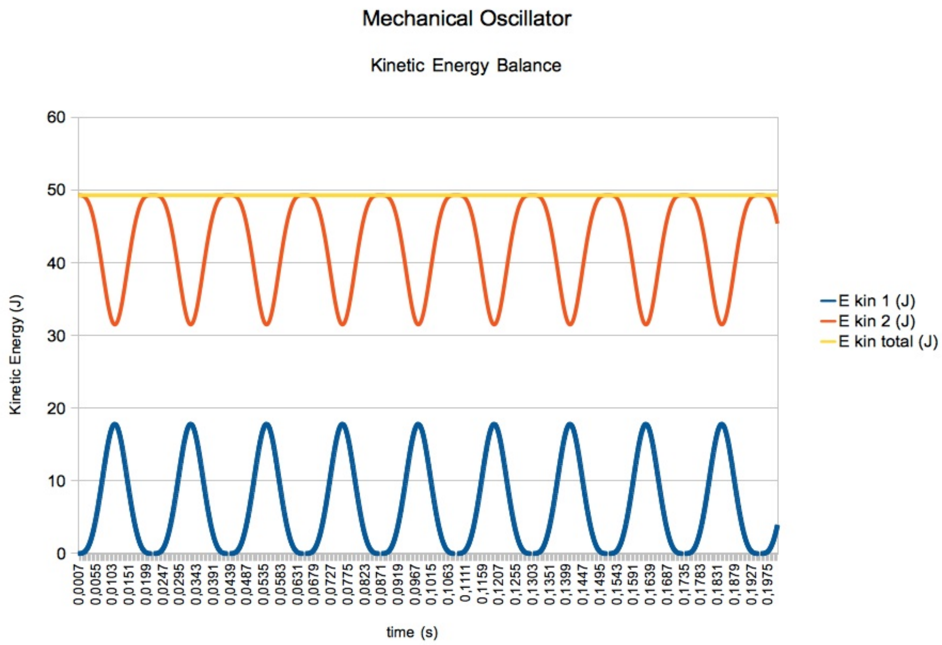

Figure 3: Kinetic energy balance.

We conclude that by setting the outer pendulum in motion with only $E_k = 50\text{J}$ we continuously either accelerate or decelerate the inner pendulum mass with a absolute power averaging $|\bar{P}_{PivotPower}| \approx 1800\text{W}$ with a maximum of $P_{MaxPivotPower} \approx 3500\text{W}$. What we want to emphasize here, is that this is reactive power. Half of the time it is decelerating the mass, i.e. working against velocity.

2 Adding Mechanical Load to the Model

We now assume that the power that manifests itself as a oscillating torque at the fixed pivot can be useful if we put a load on it, for example with a generator with resistive load. In theory all of the pivot torque could be useful, but in reality it is probably not the case because of design and construction issues. Thus the distinction between the useful m_{load} and $m_{machine}$ in the schematic below.

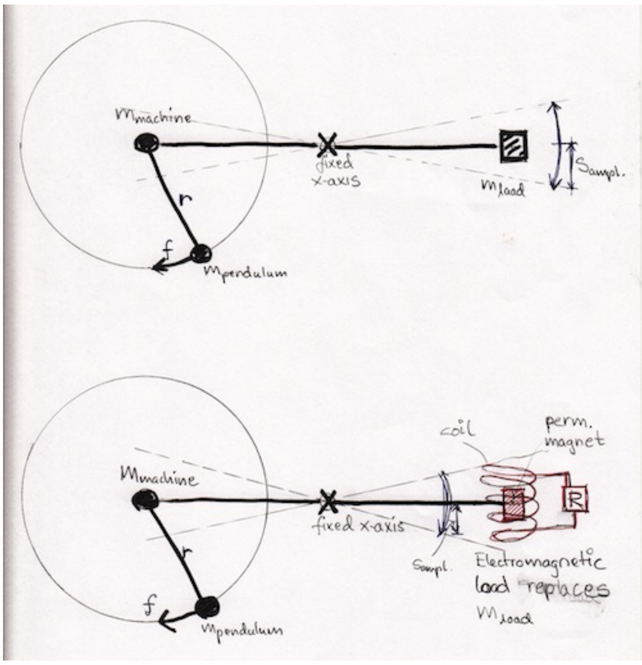

Figure 4: Double Pendulum with extra load (mass or electromagnetic)

As we mentioned above, we can only extract work against the direction of velocity. One way is to define the load as a counter velocity torque. In our model we work with a load component defined as linear function of inertia (i.e. velocity if the mass and radius are constant).

$$\tau_{load} = -\mu(m_1 + m_2)l_1\dot{\theta}_1 \tag{8}$$

μ is a factor of inertia (more or less friction), i.e. $(\frac{Nms}{kg})$. We have used this model to analyze how friction affects the system. With load as a linear function of velocity we define the rate of deceleration this load has on the system.

$$\ddot{\theta}_{1(load)} = \frac{\tau_{load}}{(m_1 + m_2)l_1^2} = -\mu\frac{\dot{\theta}_1}{l_1} \tag{9}$$

At the end of each timestep in our simulation we compensate for the load with by altering the acceleration.

$$\ddot{\theta}_{1(WithLoad)} = \ddot{\theta}_1 + \ddot{\theta}_{1(load)} \tag{10}$$

The limitations of the power extracted, is of course the kinetic energy of the inner pendulum at the time. If the load is to high the pendulum will stop (or actually oscillate each timestep in the simulation), and there will be no more energy to extract. The outer pendulum will keep rotating, but that will be of no good to us at that point.

3 Simulations with Mechanical Load

As we showed above we have an oscillating torque and velocity at the fixed pivot point. These are about 90 degrees out of phase which is shown in figure 2 above.

What we want to do now, is to add load to the system. The load is a linear function of the velocity and can be viewed as large friction. We design the load as a factor of inertia, where $\mu = 0$ Nms/kg is no load and $\mu = 1$ Nms/kg, is equivalent to a load with equal amount of inertia (think about it as mass, since all other parameters are the same). Below is a simulation with $\mu = 5$ Nms/kg.

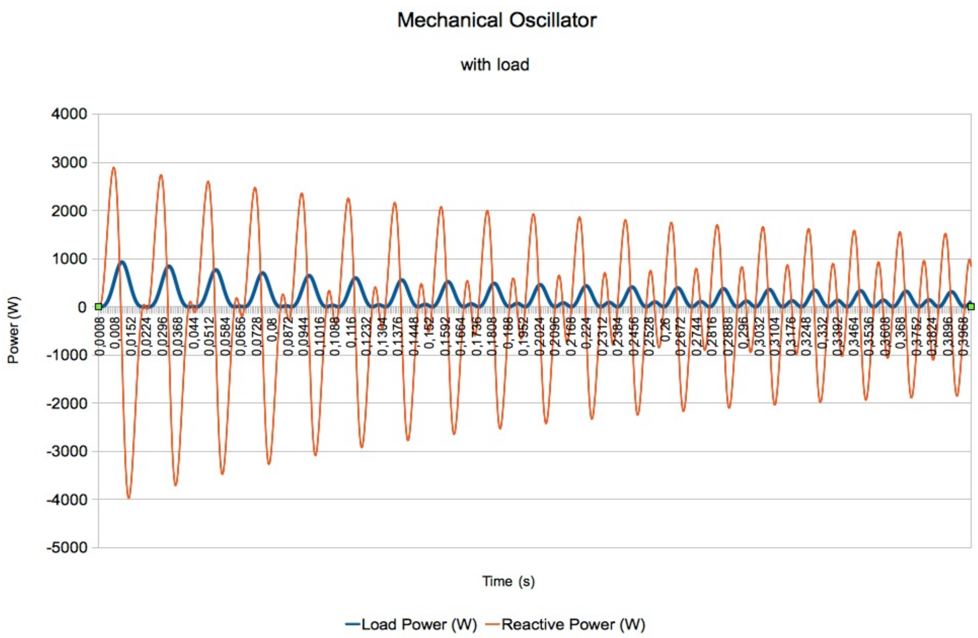

Figure 5: Reactive power and load

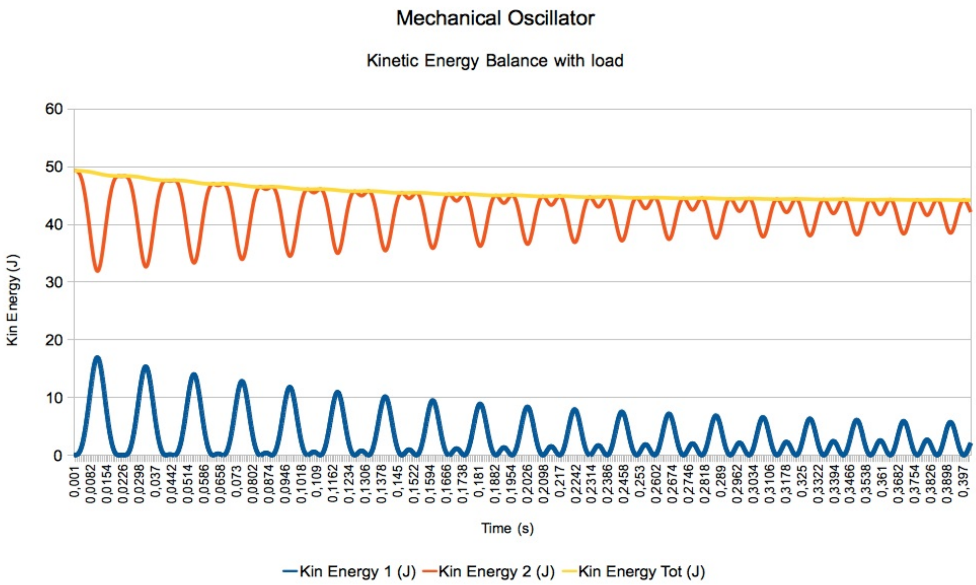

Figure 6: Kinetic energy balance with $\mu = 5$ Nms/kg load.

Below we show energy balance of the system with friction added on the outer pendulum. Note that the load we add here is only a tiny fraction (1/250) of the load added above on the inner pendulum. This is the effect of secondary oscillation of the double pendulum.

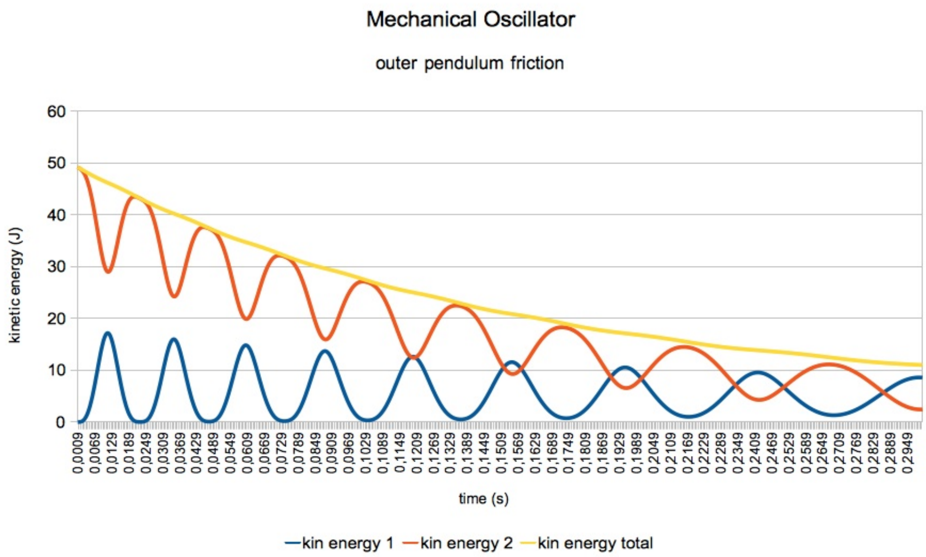

Figure 7: Kinetic energy balance with $\mu = 0.02$ Nms/kg friction load.

There are a number of interesting observations to be made in the simulation above.

- The average load is 137 W during the first second and about 106 W during the first minute.
- Kinetic energy is lost from the system during stabilization, but even after one minute only about 6 J is lost.
- If the same load is applied on the outer pendulum the pendulum would loose more than 99% of its kinetic energy within 0.03 s. With $\mu = 0.02$ Nms/kg friction load 78% of the kinetic energy is lost within 0.3s as can be seen above.
- The system will stabilize and no more kinetic energy is lost
- The load output will stabilize as well as the reactive power.
- The Reactive Power will be net-negative with about (-)100 W.

This is of course fascinating, so we experiment by putting different load factors on the system and simulate 10s, with a 0.5 ms timestep.

μ(Nms/kg)	Avg.Load(W)	React.Avg(W)	Min(W)	Max(W)
0	0	0	-3523	3523
1	24	-22	-3604	3378
5	110	-100	-3968	2895
10	216	-197	-4409	2370
20	428	-392	-5199	913
70	1449	-1328	-7813	454
150	2734	-2507	-9011	208
300	3702	-3395	-8194	78
350	3733	-3423	-7749	58
400	3688	-3381	-7296	44
600	3199	-2933	-5733	17
1000	2253	-2066	-3817	4

As we can see, we can put tremendous load on the mechanical oscillator. The Load that is possible to extract peaks about 3700 W, which incidentally is the about the same as the peak Pivot Power without load. When we reach the peak output the system is resonating and all reactive power is negative. This means that all reactive power is decelerating the pendulum and all available accelerating power is extracted by the load. There is no more power to extract, and the power starts to drop.

The COP will be almost infinite; after 60s only 14 J is lost generating about 187 kJ.

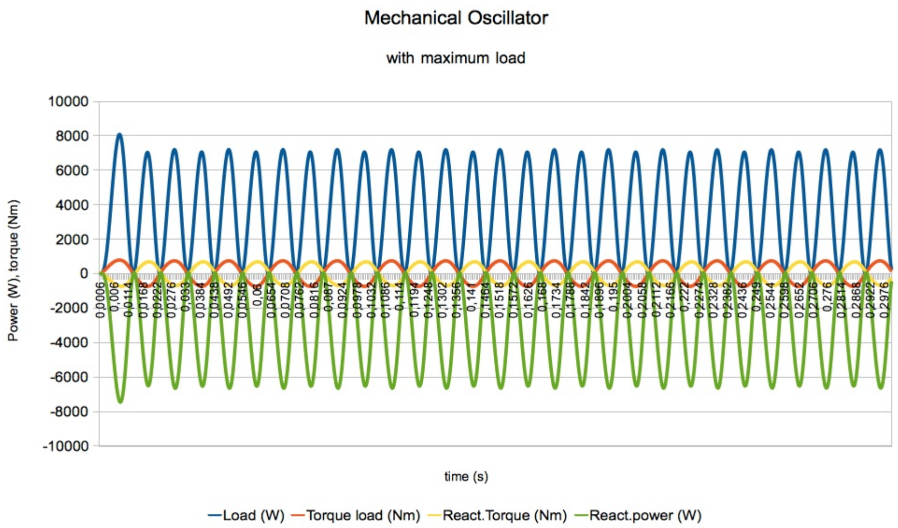

Figure 8: Mechanical Oscillator with maximum load

4 The Case of the Milkovic Pendulum

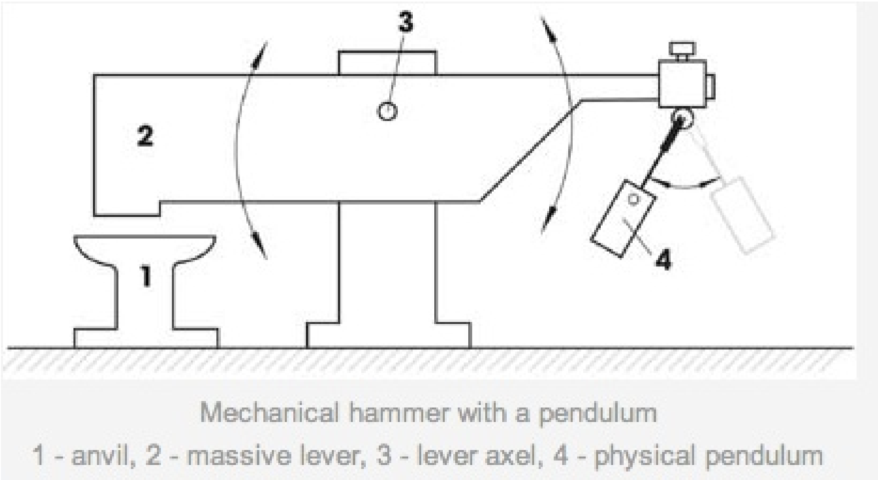

Figure 9: Milkovic Pendulum (from http://www.pendulum-lever.com)

The Milkovic pendulum is a handdriven pendulum that could be used to pump water. We've been examining a pendulum with our numerical model with the following input parameters related to figure 1: $m_1 = 50$kg, $m_2 = 10$kg, $l_1 = 0.5$m, $l_2 = 0.3$m, $\theta_2 = 2$rad (i.e. we lift the outer pendulum before we let it go, thereafter only affected by gravity).

Lets have a look at the characteristics of the movements, energy balance and reactive power of the pendulum in a frictionless environment.

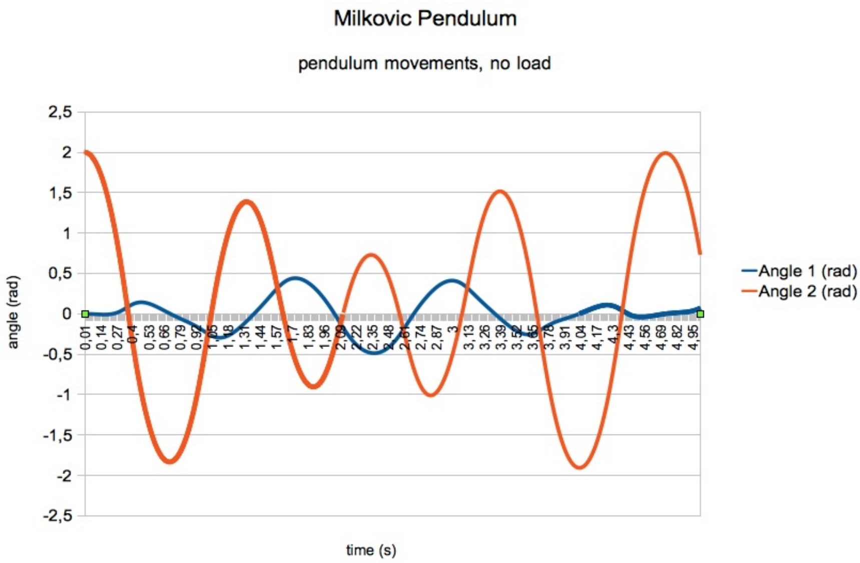

Figure 10: Pendulum movements

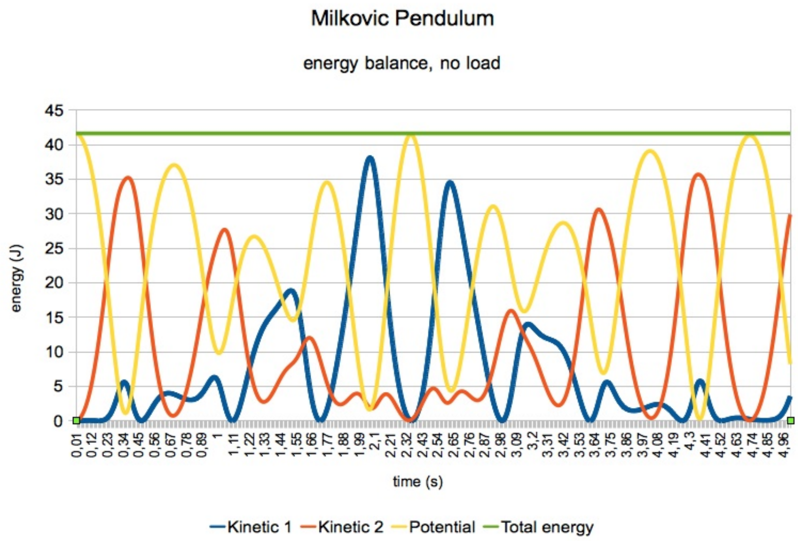

Figure 11: Energy balance

The exchange of kinetic and potential energy is quite complex, but the total energy of the system is constant.

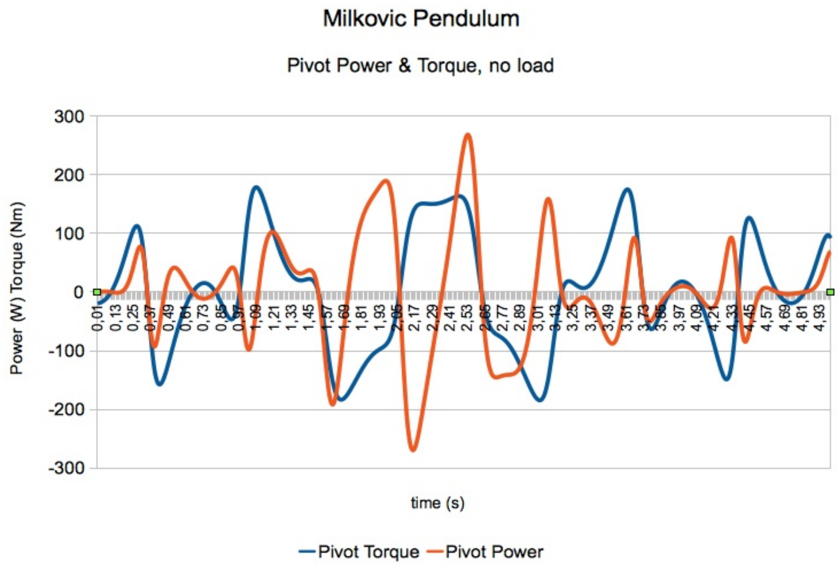

Figure 12: Reactive power and torque at the fixed pivot.

Now, lets try to simulate the Milkovic pendulum above with a load of $\mu =$ 10Nms/kg. That will equal a peak torque of about 200 Nm, which is quite reasonable for a handdriven pump.

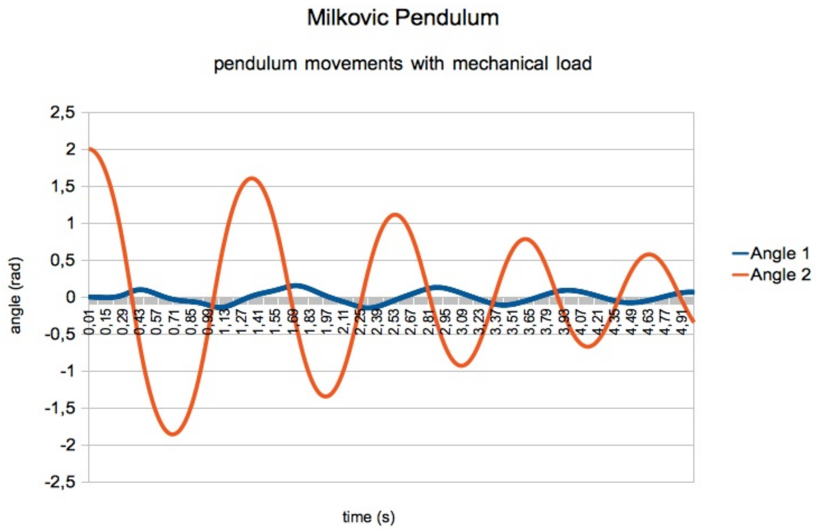

Figure 13: Pendulum movements with load, $\mu = 10$ Nms/kg

As we can see, the inner pendulum movements (angle 1) is rather small in relation to the outer pendulum. At the start of the simulation the amplitude is about 0.24 rad, which with a radius of 0.5 m is 12 cm. This also seems usable for a handdriven pump, maybe with some leverage.

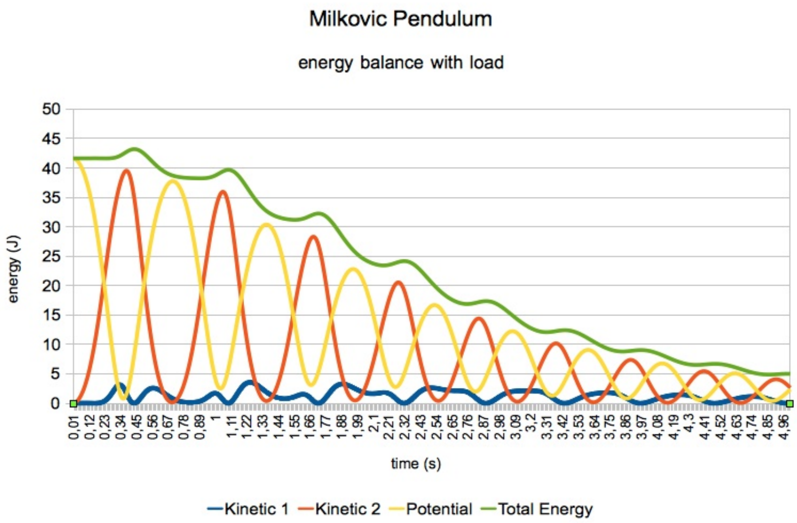

Figure 14: Energy balance of Milkovic pendulum with load

The pendulum has some really strange characteristics when the acceleration and velocity are not aligned (i.e. different signs). We are actually adding more kinetic energy to the outer pendulum than we are extracting by loading the inner pendulum. The load actually accelerates the outer pendulum more than it reduces velocity of the inner one. So at brief moments of time total energy is actually increasing even though we load the system.

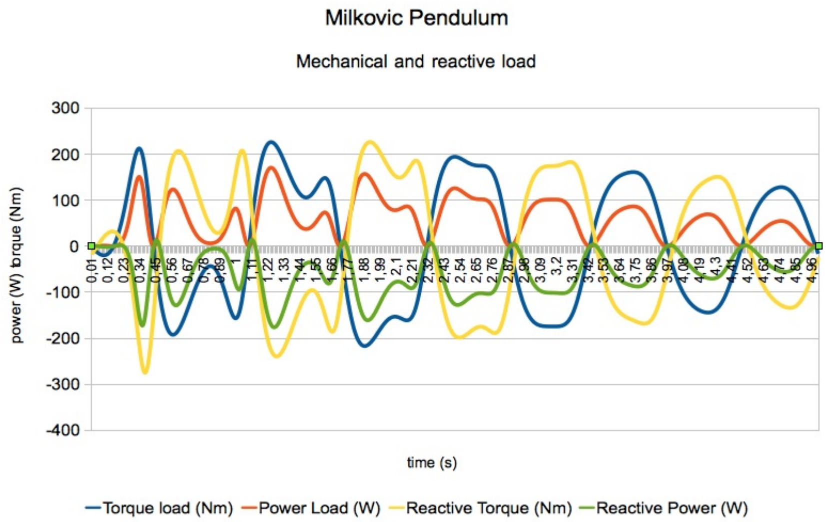

Figure 15: Power and torque extracted from a Milkovic Pendulum

The pendulum outputs on average about 60W with no input except for the $E_{in} = 42$ J used to lift the pendulum initially (2 rad). After 5 seconds the output has dropped to about half due to the frictional aspects of the load and 37J of kinetic energy is lost. However during these 5 seconds a total of 290J are extracted which equals a COP of 7.8.

If we have a closer look at the graphs we can se that about 17 J are lost during the first two seconds (i.e. 3 oscillations), which means that if we replace these, the pendulum will keep swinging with constant amplitude. This is also the aim of pushing the pendulum-pump.

The result is that by adding 8.5 W of input power we continuously generate about 60 W of output power. A COP of about 7.

Our conclusion is that these simulations are very consistent with the experimental results achieved by Milkovic. However, this could of course be examined in much more detail.

5 Generating Power

This report is however not so interested in handdriven pendulums as in the possibility to build a generator that utilize the power in a automized manner and generating electrical power at higher frequencies.

The interdependencies between the two pendulums continuously exchanging kinetic energy as they oscillate is complex. The two pendulums transfer kinetic energy between themselves in both directions as described by the Euler Lagrange equations earlier.

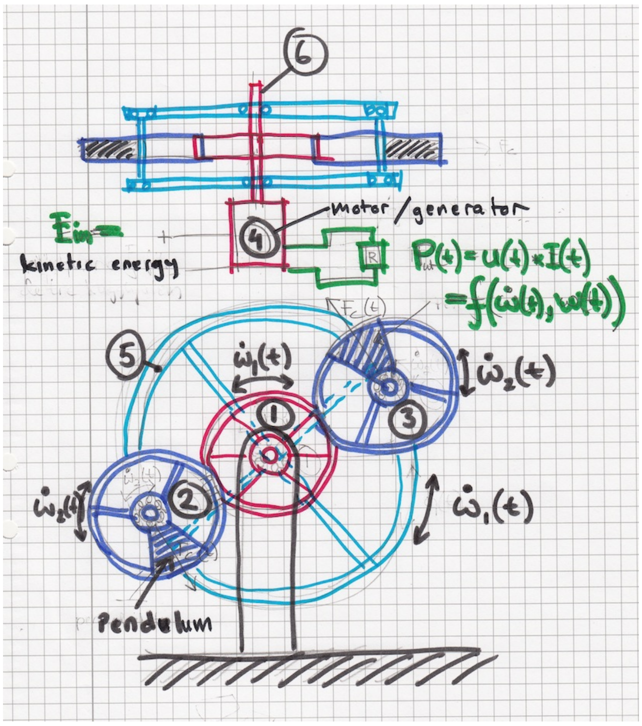

Figure 16: Construction schematic

This is a simple schematic of a generator using what we've described. On the input side we have a motor attached to a driving cogwheel, which rotates the two unbalanced cogwheels (i.e. pendulums). These cogwheels are synchronized so that all forces in the direction to and from the fixed axis is cancelled out at all times, which will reduce the stress on the fixture. The unbalanced cogwheels are mounted on a frame that transfers momentum to the generator.

- (1), (2) and (3) are cogwheels, where (2) and (3) are unbalanced. They can be considered pendulums.
- (4) is a motor/generator

- (2) and (3) are mounted on a "frame" (5) which rotates frictionless on the axis (6).
- The motor/generator is fixed on cogwheel (1) so that it can either drive or receive energy/momentum to/from the system.
- The motor/generator is securely mounted on the floor/surroundings.

We then suggest the following line of thought.

1. The frame (5) with the cogwheels (2) and (3) is fixed so that it can not rotate around axis (6).
2. The cogwheels are set in motion at the desired rotation speed with the motor. The frame is still fixed.
3. The motor drive is disconnected, now the motor acts as a generator. Still there is no load so the generator rotates frictionless.
4. The frame (5) fixation is disconnected sot that it now rotates/oscillates freely around axis (6) and transfers momentum to the generator (4). This might be technicaly challenging and maybe separation between motor and generator might be preferable.
5. Load is added on the generator.

If the machine was frictionless we would be able to extract power continuously. However this is of course not the case and a system for adding power to overcome friction is needed. This is the tricky part because there is a constant feedback of momentum coming from the pendulums (2) and (3) to the driving cogwheel (1). This means that it will not be possible to add a constant driving momentum on cogwheel (1) since this momentum simply will work the machine in the wrong direction half of the time.

If we try to force the outer pendulum into a certain velocity, we will no doubt disturb the inner pendulum as it transfer momentum back. Constant rotation of the outer pendulum will simply not do. Therefore, we need to design a system that pulses the correct amount of energy/momentum to the driving cogwheel (1) with the right timing. Obviously this would be in the opposite direction of friction, i.e. always in the direction of the current velocity and acceleration. It's actually a simple case of resonance. The analogy for this is obvious; like pushing a playground swing, or for that matter, the Milkovic pendulum.

6 Building a Household Generator

If we, as an example, want to build a household generator we could use two 0.1 kg pendulums with a radius of 0.1 m, mounted on a frame with the radius 0.2 m. The pendulums are set in initial rotation at 50Hz. If the weight of the cogwheels and levers are assumed to be positioned at the cogwheel/pendulum center and is 1 kg per pendulum, the output will be approximately 7.2 kW of AC power. The output power distribution (reactive and resistive) and electrical output over a 10 ohm load, will look as follows.

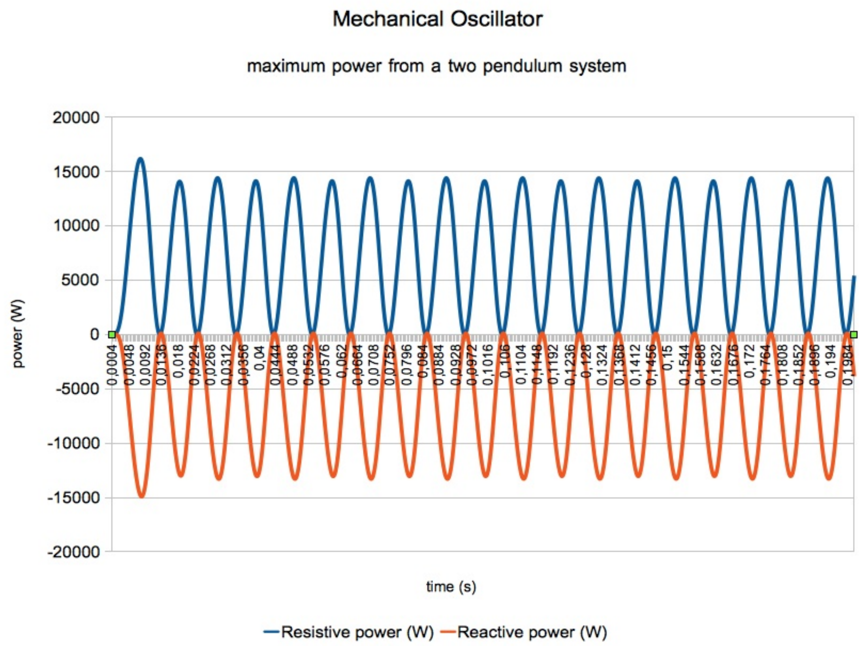

Figure 17: Resistive and reactive power distribution

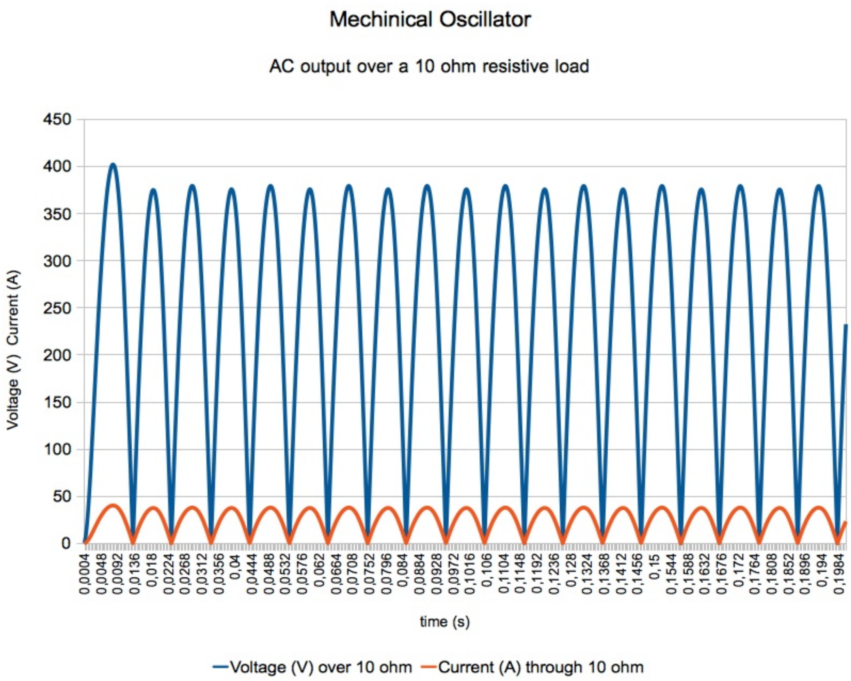

Figure 18: Electical AC output schematic with 10 ohm resistive load

The equations for the output power is as follows. $m_{MechLoad}$ describes the productive load/momentum, which is not the same as m_1 that is the combined mass of the load and the unproductive weight of the cogwheels and frame (i.e. machine).

$$P(t)_{output} = m_{MechLoad} r_{frame}^2 \omega'_{AccOfFrame} \omega_{SpeedOfFrame} \tag{11}$$

From Kirchoff we know that the voltage over a coil and a resistive load is described as follows.

$$Li' + iR_{load} = 0 \tag{12}$$

Consequently we get for the complete generator with electrical output.

$$P(t)_{output} = Lii' = i^2 R_{load} = m_{MechLoad} r_{frame}^2 \omega'_{AccOfFrame} \omega_{SpeedOfFrame} \tag{13}$$

The current i_{output} correlates directly with the speed of rotation ω and the voltage $u_{output} = Li'$ correlates directly with the acceleration ω'. And with maximum load these are now in phase. We got pure AC power over our resistive load.

7 Conclusions

This document shows that it is possible to utilize the constant force acting through the arm of a pendulum in motion. The force is a function of speed of rotation (ω) but results in an acceleration of mass, i.e energy. The energy will manifest itself in the form of oscillations. We then make the connection between these oscillations and the characteristics of AC current and realize that it is exactly what we are looking for.

The power extracted is a function of the frequency of rotation by the power of three ($mr^2\omega'\omega$, Lii' or Ri^2). This also explains the extreme power of vibrations, for example in buildings, bridges and other constructions. Even minute imbalances in an engine gets the whole car to vibrate. And so on, and so on.

At this point we would also suggest this method of analysis to examine different solid state setups using transformers, coils and capacitors in combination with AC or DC motor/generators and batteries.

We also show the theory behind the dynamics can be described using a numerical solution (Runge Kutta) of the Euler Lagrange equations of the double pendulum with added mechanical and resistive load. The centrifugal force of the rotating/oscillating outer pendulum creates a property that can be described as an *artificial gravity*, and it can be utilized using the method described.

With this foundation of controlled oscillations directly converted into AC power, simple motors can be built in any small village workshop everywhere around the globe. Help can be supplied with construction and motor design if needed. We emphasize the importance of design, and that the construction for transfer of power need to be lightweight i relation to the weight of the pendulum.

As our hero Nikola Tesla famously said: *"If you want to find the secrets of the universe, think in terms of energy, frequency and vibration."*